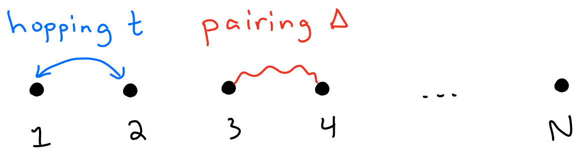
Kitaev Chain

Take spinless fermions on N-site chain w/ PBC's (for now)



(Pauli blocks on-site pairing)

$$\boxed{H = \sum_x \left[-\mu c_x^\dagger c_x - \frac{1}{2}\left(t\, c_x^\dagger c_{x+1} - \Delta\, c_x^\dagger c_{x+1}^\dagger + h.c. \right) \right]}$$

IR

PBC's ⇒ go to k-space,

$$\boxed{H = \sum_{k \in BZ} \left[\xi_k c_k^\dagger c_k + \frac{1}{2}\left(\Delta_k c_k^\dagger c_{-k}^\dagger + h.c. \right) \right]}$$

with

$$\xi_k = -t\cos k - \mu$$

$$\Delta_k = i\Delta \sin k$$

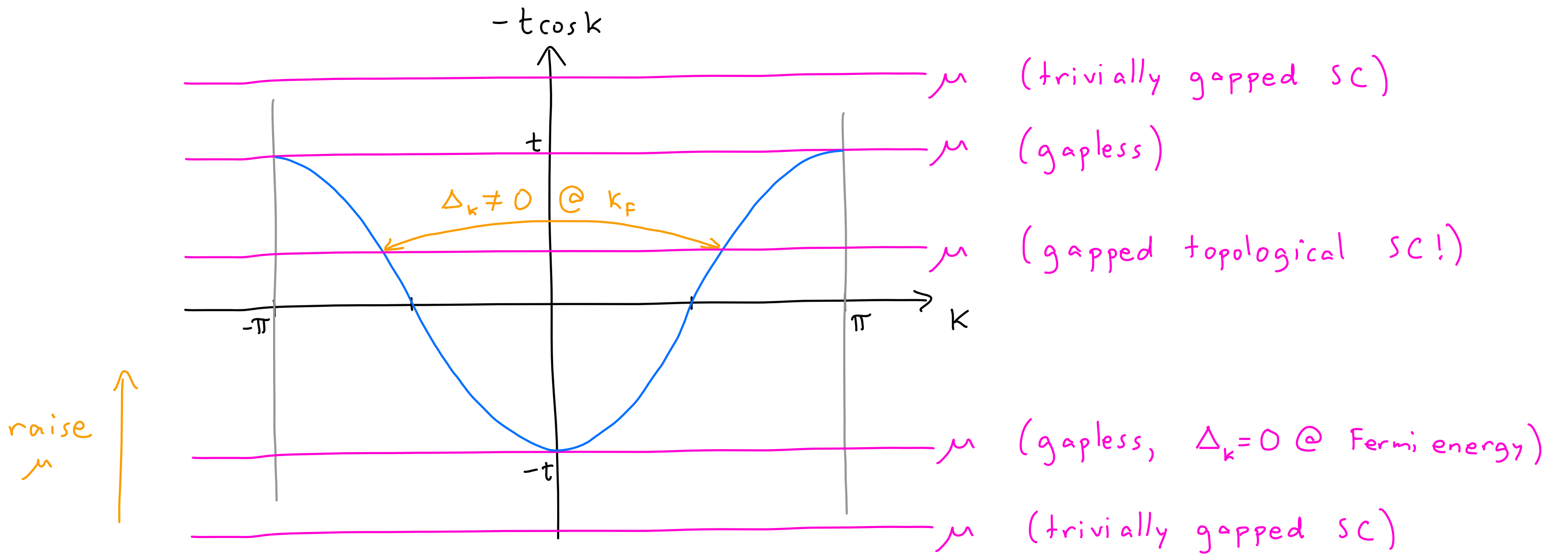
odd parity forced by spinlessness

Excitation energy is

$$\boxed{E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}}$$

Can now deduce phase diagram.

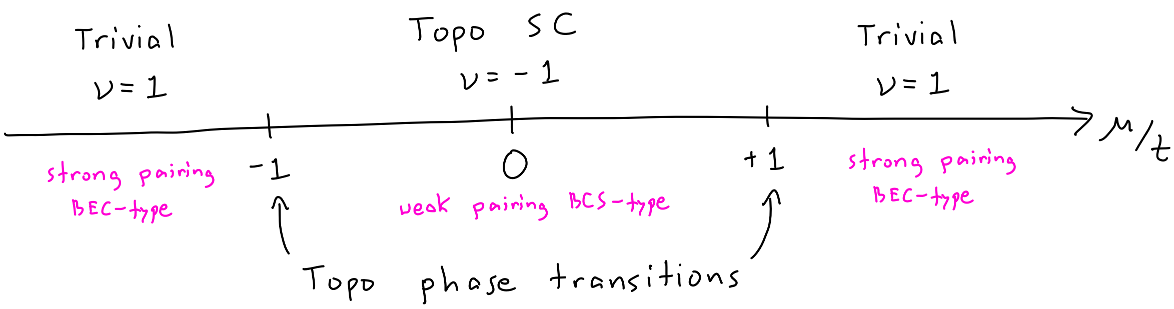
Phase Diagram



$\mathbb{Z}_2$ topological invariant distinguishing gapped SC's is

$$\boxed{\nu = (-1)^{\#\text{pairs of Fermi pts.}}}$$

Invariant can differ if enforcing symmetries, e.g., $\mathbb{Z}$ invariant w/ $\mathcal{T}^2 = +1$ time reversal.



Want to explore universal properties of phases/transitions. Convenient to take $\boxed{\Delta = t}$ hereafter.

hopping

$$\Rightarrow \boxed{H = \sum_x \left[-\mu c_x^\dagger c_x - \frac{1}{2} t \left(c_x^\dagger + c_x\right)\left(c_{x+1} - c_{x+1}^\dagger\right) \right]}$$

pairing

Gapped SC's

Familiar also from Kitaev honeycomb model

Take open BC's now + use Majorana rep:

$$\boxed{c_x = \frac{1}{2}\left(\gamma_{Bx} + i\gamma_{Ax}\right)}$$

Majorana ops.; akin to $z = a + ib$

$$\boxed{\gamma_\alpha^{\dagger} = \gamma_\alpha\,,\ \gamma_\alpha^2 = I\,,\ \{\gamma_\alpha, \gamma_{\alpha' \neq \alpha}\} = 0}$$

Majorana op. algebra; reproduces $c_x^2 = (c_x^\dagger)^2 = 0$, $\{c_x, c_{x'}^\dagger\} = \delta_{xx'}$

Rewrite H:

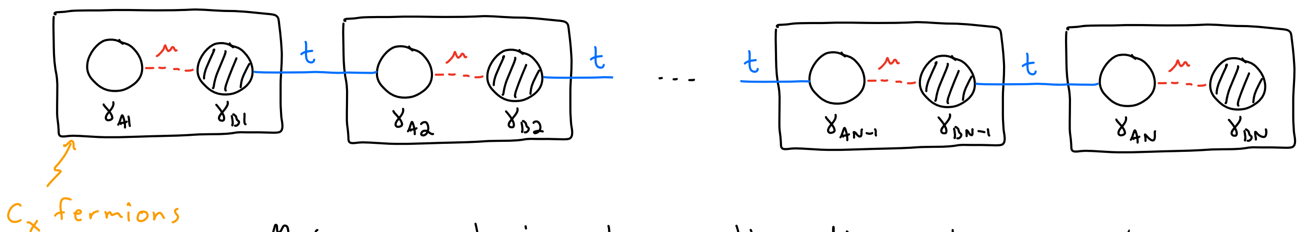
$$c_x^\dagger c_x = \frac{1}{4}\left(\gamma_{Bx} - i\gamma_{Ax}\right)\left(\gamma_{Bx} + i\gamma_{Ax}\right) = \frac{1}{4}\left(1 + 1 + i\gamma_{Bx}\gamma_{Ax} - i\gamma_{Ax}\gamma_{Bx}\right)$$

$$= \frac{1}{2}\left(1 + i\underbrace{\gamma_{Bx}\gamma_{Ax}}_{=\pm 1}\right)$$

$$\left(c_x^\dagger + c_x\right)\left(c_{x+1} - c_{x+1}^\dagger\right) = i\gamma_{Bx}\gamma_{Ax+1}$$

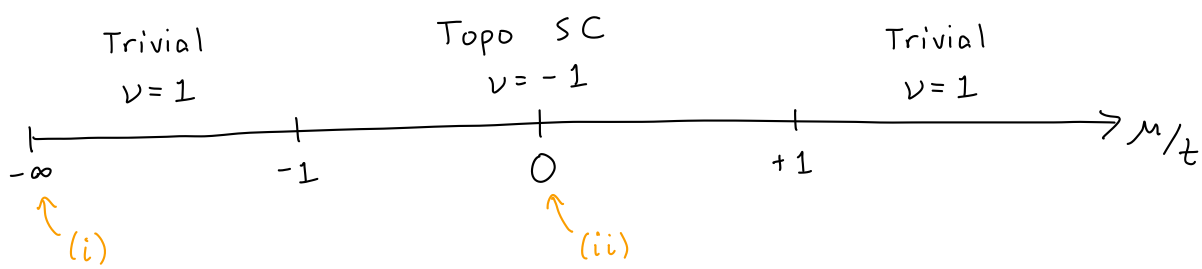
$$\Rightarrow \boxed{\boxed{H = -\frac{i}{2}\sum_x \left(\mu\,\gamma_{Bx}\gamma_{Ax} + t\,\gamma_{Bx}\gamma_{Ax+1}\right)}}$$

dropping const

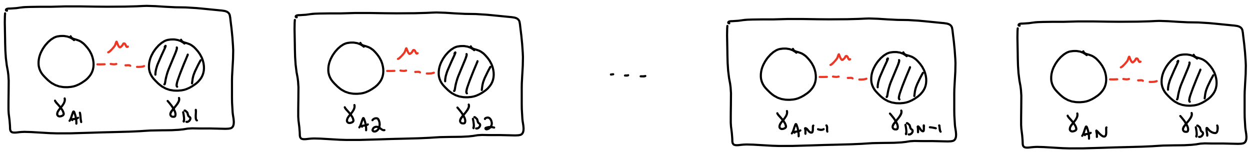


Majorana chain w/ competing dimerizations μ, t!

For revealing snapshots of gapped phases, examine 2 limits:



(i) $\underline{\mu < 0,\ t = 0}$

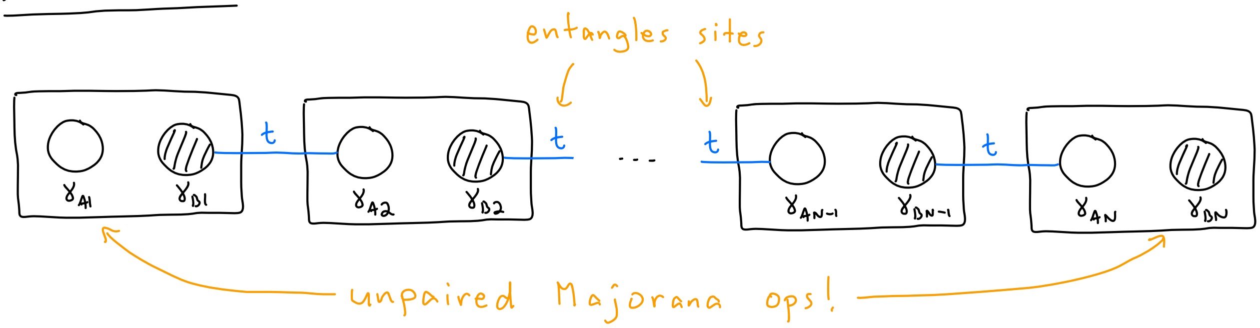


Unique gd. state w/ no entanglement between sites,

$$\boxed{|\psi\rangle = |\text{vac of } c_x \text{ fermions}\rangle}$$ ↜ equivalently, state w/ $i\gamma_{Bx}\gamma_{Ax} = -1 \ \forall\ x$

Gap μ to add a c_x fermion ↜ equivalently, to flip $i\gamma_{Bx}\gamma_{Ax} \to +1$

(ii) $\underline{\mu = 0,\ t > 0}$



Define new set of fermions,

$$f_x = \frac{1}{2}\left(\gamma_{Ax+1} + i\gamma_{Bx}\right) \quad \Rightarrow \quad f_x^+ f_x = \frac{1}{2}\left(1 - i\gamma_{Bx}\gamma_{Ax+1}\right)$$

$$\gamma_1 \equiv \gamma_{A1}, \quad \gamma_2 \equiv \gamma_{BN}$$ ↜ unpaired Majoranas

$$d = \frac{1}{2}\left(\gamma_1 + i\gamma_2\right)$$ ↜ <u>non-local</u> canonical fermion op

$$\Rightarrow \quad \boxed{H = t \sum_x f_x^+ f_x + 0 \times d^+ d}$$ ↜ dropping const

f_x vacuum minimizes energy; bulk gap t for adding f_x fermion

But can add/remove d fermion w/ no energy cost!

So two-fold (topological) gd state degeneracy:

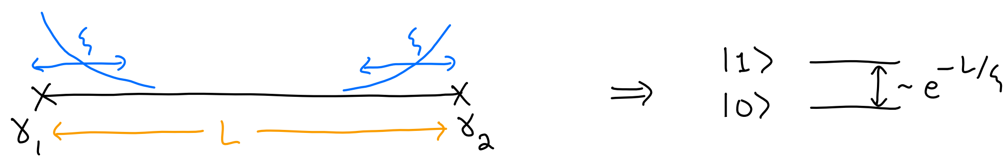
$$\boxed{\begin{aligned} |0\rangle &= |\text{vac of } f_x, d \text{ fermions}\rangle \\ |1\rangle &= d^\dagger |0\rangle \end{aligned}}$$

States carry opposite fermion parity — "Majorana zero modes" $\gamma_{1,2}$ remove energy cost for unpaired e^-

Comments:

- Edge zero modes encoded in entanglement of <u>periodic</u> chain!

- For $\mu \neq 0$, $t \neq \Delta$, zero modes decay exponentially into bulk



with $\xi \sim \dfrac{\hbar v_F}{E_{gap}}$

- At $L \gg \xi$, degeneracy (modulo exp. small splitting) is immune to arbitrary local perturbations (sufficiently weak to preserve topo phase). Why?

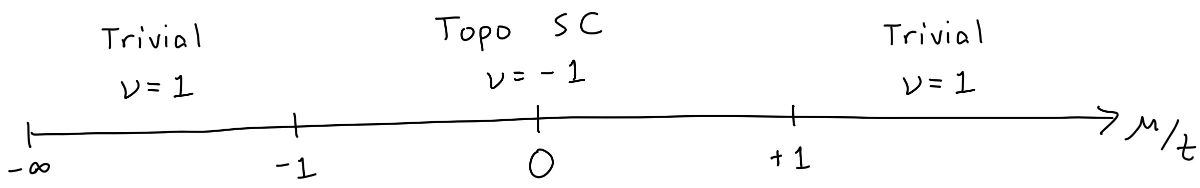
$$\delta H = i \frac{\delta\epsilon}{2} \gamma_1 \gamma_2$$ ← low-energy op that splits deg by $\delta\epsilon$

But spatial separation of $\gamma_{1,2}$ implies any local pert. gives $\delta\epsilon \sim e^{-L/\xi}$!

- Corollary: local measurements can't reveal which gd. state system realizes! Detecting $d^\dagger d$ requires non-locally probing γ_1 <u>and</u> γ_2 to infer $i\gamma_1\gamma_2 = \pm 1$. ← related to global fermion parity, which clearly can't be determined via local measurement!

Last 2 properties appealing for quantum info storage (more later).

Topo Phase Transition



Focus here + take $\boxed{\mu = -t}$

Low-energy physics @ criticality?

$$H \to H_{crit} = -\frac{it}{2}\sum_x \left(-\gamma_{Bx}\gamma_{Ax} + \gamma_{Bx}\gamma_{Ax+1}\right) = -\frac{it}{2}\sum_x \gamma_{Bx}\left(\gamma_{Ax+1} - \gamma_{Ax}\right)$$

$$\underset{\text{continuum limit}}{\sim} \boxed{-\frac{it}{2}\int_x \gamma_B \partial_x \gamma_A}$$

Writing $\boxed{\gamma_{A/B} = \gamma_R \pm \gamma_L}$ gives

$$H_{crit} = -\frac{it}{2}\int_x (\gamma_R - \gamma_L)\partial_x(\gamma_R + \gamma_L)$$

$$= -\frac{it}{2}\int_x \Big(\gamma_R \partial_x \gamma_R - \gamma_L \partial_x \gamma_L + \underbrace{\gamma_R \partial_x \gamma_L - \gamma_L \partial_x \gamma_R}_{\text{cancel}}\Big)$$

So we get

$$\boxed{H_{crit} = \int_x \left(-i\hbar v \gamma_R \partial_x \gamma_R + i\hbar v \gamma_L \partial_x \gamma_L\right)} \qquad (v \propto t)$$

R/L moving gapless Majorana fermions

with $$\boxed{c_x = \frac{1}{2}\left(\gamma_{Bx} + i\gamma_{Ax}\right) \sim e^{i\frac{\pi}{4}}\gamma_R(x) - e^{-i\frac{\pi}{4}}\gamma_L(x)}$$

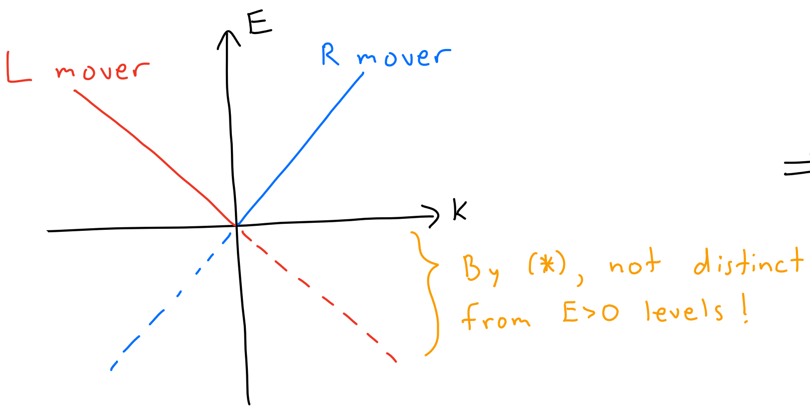
dictionary linking microscopic ops to low-energy fields

For deeper insight, take PBC + go to k-space:

$$\boxed{\gamma_{R/L}(x) = \int_k e^{ikx}\, \gamma_{R/L}(k)} \qquad \gamma_{R/L}(x) = \gamma^{\dagger}_{R/L}(x) \Rightarrow \boxed{\gamma_{R/L}(k) = \gamma^{\dagger}_{R/L}(-k)} \quad (*)$$

$$\Rightarrow \boxed{H_{crit} = \int_k \hbar v k \left[\gamma^{\dagger}_R(k)\gamma_R(k) - \gamma^{\dagger}_L(k)\gamma_L(k)\right]}$$

← Can write in terms of only E>0 modes using (*)

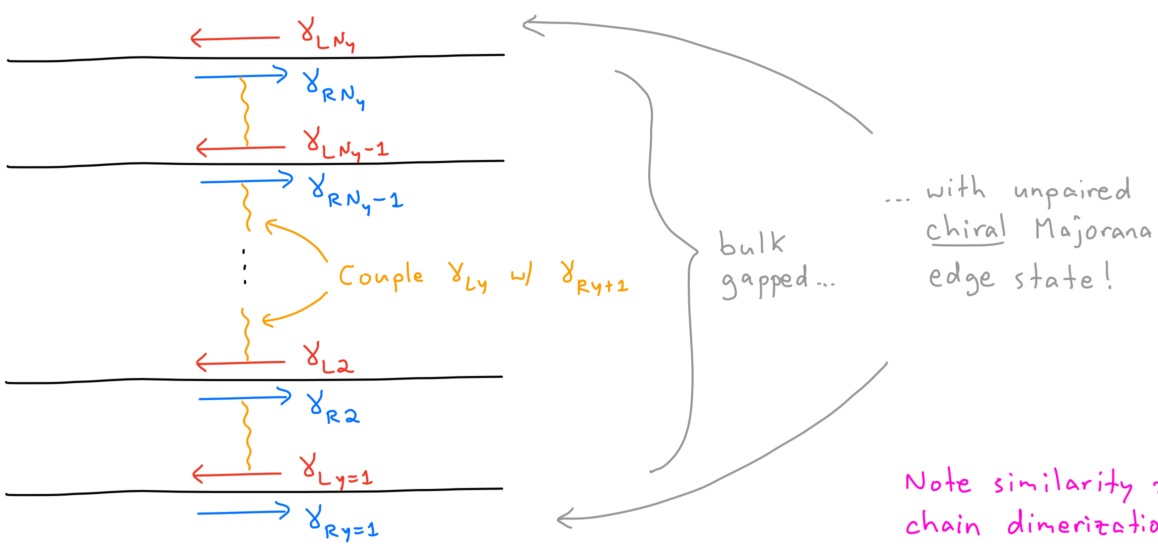


$\Rightarrow$ $\boxed{\text{"half" of single-channel wire}}$

Next, we'll leverage Kitaev chains to access...

2D Topological SC's

Consider an array of critical Kitaev chains, indexed by $y = 1, \dots, N_y$:



Note similarity to Kitaev chain dimerization

Aside I: Effective Hamiltonian for coupled pair $\gamma_{Ly}, \gamma_{Ry+1}$,

$$H_{y,y+1} = \int_x \left(-i\hbar v\, \gamma_{Ry+1}\partial_x \gamma_{Ry+1} + i\hbar v\, \gamma_{Ly}\partial_x\gamma_{Ly} + 2im\,\gamma_{Ly}\gamma_{Ry+1}\right)$$

go to k-space

$$= \int_k \left[\gamma^\dagger_{Ry+1}(k) \quad \gamma^\dagger_{Ly}(k)\right] \underbrace{\begin{bmatrix} \hbar v k & -im \\ im & -\hbar v k \end{bmatrix}}_{\text{Energies are } \sqrt{(\hbar v k)^2 + m^2}} \begin{bmatrix} \gamma_{Ry+1}(k) \\ \gamma_{Ly}(k) \end{bmatrix}$$

$\leftarrow$ gapped!

Aside II: Origin of inter-chain coupling? Use dictionary

$$\boxed{c_{x,y} \sim e^{i\frac{\pi}{4}}\gamma_{Ry}(x) - e^{-i\frac{\pi}{4}}\gamma_{Ly}(x)}$$

to get

$$\boxed{\begin{aligned} H_{inter} &= t'\sum_{x,y}\Big(\underbrace{e^{i\frac{\pi}{4}}c_{x,y} + e^{-i\frac{\pi}{4}}c^\dagger_{x,y}}_{\sim\gamma_{Ly}}\Big)\Big(\underbrace{e^{i\frac{\pi}{4}}c_{x,y+1} - e^{-i\frac{\pi}{4}}c^\dagger_{x,y+1}}_{\sim i\gamma_{Ry+1}}\Big) \\ &= t'\sum_{x,y}\left(c^\dagger_{x,y}c_{x,y+1} + i\,c^\dagger_{x,y}c^\dagger_{x,y+1} + h.c.\right)\end{aligned}}$$

hopping

pairing w/ phase i relative to intra-chain pairing

$\therefore$ Chiral 2D topo SC $\sim$ weak pairing spinless $\underbrace{p_x + ip_y}$ SC

pairing pot. odd in x,y directions, with rel. phase i

Aside III: Real-space, k-space parities are related:

$$\sum_{\vec r}\sum_{\delta\vec r}\Delta(\delta\vec r)\, c^\dagger_{\vec r} c^\dagger_{\vec r+\delta\vec r} = \sum_{\vec r}\sum_{\delta\vec r}\Delta(-\delta\vec r)\, c^\dagger_{\vec r} c^\dagger_{\vec r-\delta\vec r} = \sum_{\vec r}\sum_{\delta\vec r}\Delta(-\delta\vec r)\, c^\dagger_{\vec r+\delta\vec r} c^\dagger_{\vec r}$$

$$= -\sum_{\vec r}\sum_{\delta\vec r}\Delta(-\delta\vec r)\, c^\dagger_{\vec r} c^\dagger_{\vec r+\delta\vec r} \Rightarrow \boxed{\Delta(\delta\vec r) = -\Delta(-\delta\vec r)} \leftarrow \text{odd}$$

$$= \frac{1}{N}\sum_{\vec k,\vec k'\in BZ}\sum_{\vec r}\sum_{\delta\vec r}\Delta(\delta\vec r)\, e^{-i\vec k\cdot\vec r} e^{-i\vec k'\cdot(\vec r+\delta\vec r)}\, c^\dagger_{\vec k} c^\dagger_{\vec k'} = \sum_{\vec k\in BZ}\sum_{\delta\vec r}\Delta(\delta\vec r)\, e^{i\vec k\cdot\delta\vec r}\, c^\dagger_{\vec k} c^\dagger_{-\vec k}$$

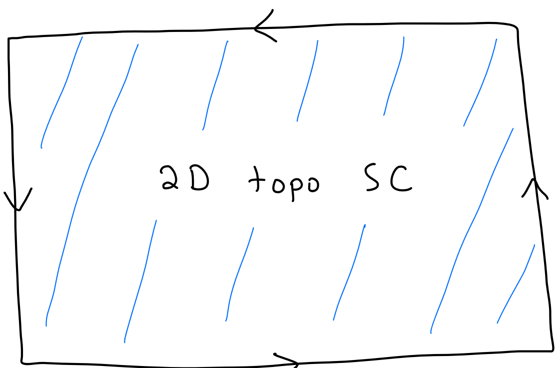
$$\equiv \sum_{\vec k\in BZ}\Delta(\vec k)\, c^\dagger_{\vec k} c^\dagger_{-\vec k} \quad \text{with} \quad \boxed{\Delta(\vec k) = \sum_{\delta\vec r}\Delta(\delta\vec r)\, e^{i\vec k\cdot\delta\vec r}}$$

$$\Delta(\vec k) = -\sum_{\delta\vec r}\Delta(-\delta\vec r)\, e^{i\vec k\cdot\delta\vec r} = -\sum_{\delta\vec r}\Delta(\delta\vec r)\, e^{-i\vec k\cdot\delta\vec r} = -\Delta(-\vec k)$$

oddness of $\Delta(\delta\vec r)$... ... implies oddness of $\Delta(\vec k)$

Our p_x+ip_y SC has $\Delta(\delta\vec r=\hat x)=\frac{\Delta}{2}$, $\Delta(-\hat x)=-\frac{\Delta}{2}$, $\Delta(\hat y)=\frac{i\Delta}{2}$, $\Delta(-\hat y)=-i\frac{\Delta}{2}$.

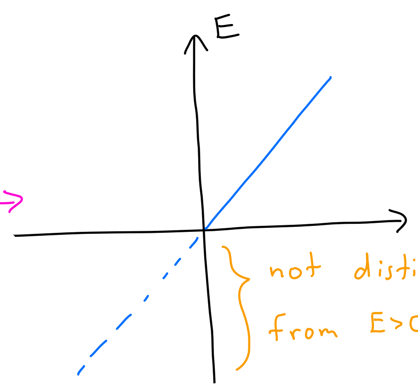
Finite planar geometry gives



chiral Majorana
$= \frac{1}{2}$ Integer QH edge state

$$\boxed{H_{edge} = \int_u \left(-i\hbar v\, \gamma\, \partial_u \gamma\right)} \Rightarrow \boxed{E = \hbar v k}$$

edge coord. chiral Majorana field



not distinct from $E>0$ levels

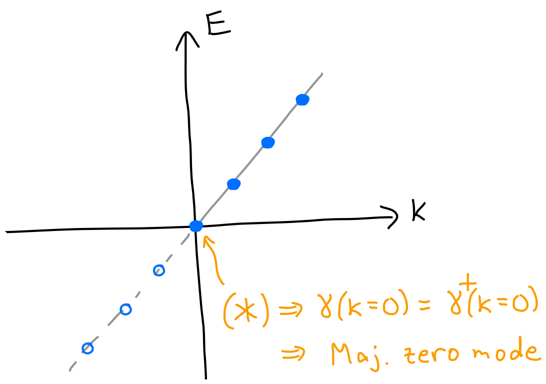
Key question: Edge spectrum for finite perimeter L_{edge}?

Only 2 options!

BC's compatible w/ $\gamma = \gamma^\dagger$

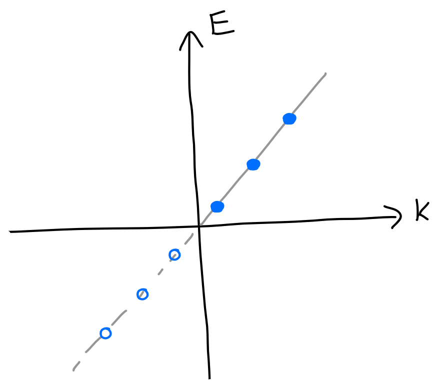
(A) PBC's $\gamma(u + L_{edge}) = \gamma(u)$

$$k \to k_n = \frac{2\pi}{L_{edge}} \times n$$



(B) anti-PBC's $\gamma(u + L_{edge}) = -\gamma(u)$

$$k \to k_n = \frac{2\pi}{L_{edge}} \times \left(n + \tfrac{1}{2}\right)$$

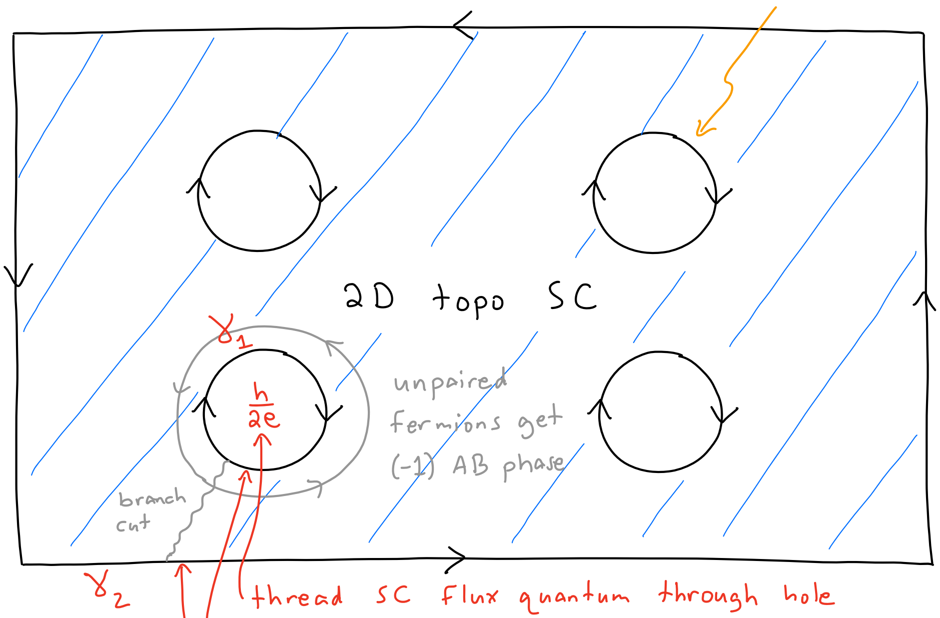


Correct answer; can't have odd # of Maj. zero modes (Hilbert space wouldn't make sense), ruling out PBC's

PBC's can, however, be activated!

Drill holes:

any boundary binds chiral Majorana



Zero modes @ other holes generated similarly

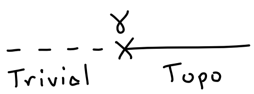
thread SC flux quantum through hole

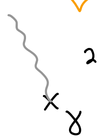
Majorana BC's shift to periodic ⇒ zero mode <u>pair</u> γ_1, γ_2!

Corollary: $\boxed{\text{2D topo SC vortices } (\sim \text{self-generated "holes"}) \text{ bind Majorana zero modes}}$

Non-Abelian Statistics

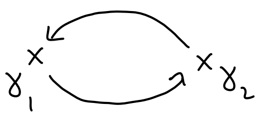
$\underbrace{\text{Domain walls}}$ + $\underbrace{\text{vortices}}$ w/ Maj. zero modes obey non-Abelian braiding!





2D Topo SC

Follows from:



swaps fermion halves!

$$d = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

$$d^\dagger d = \frac{1}{2}(1 + i\gamma_1\gamma_2)$$

Invariant under braid due to global fermion parity conservation

$$\Rightarrow \boxed{\begin{matrix} \gamma_1 \to \gamma_2 \\ \gamma_2 \to -\gamma_1 \end{matrix}}$$

[Or $\begin{matrix} \gamma_1 \to -\gamma_2 \\ \gamma_2 \to \gamma_1 \end{matrix}$; minus is intuitive from branch cuts in vortex case]

Implemented by

$$\boxed{U_{12} = e^{i\frac{\pi}{4}\underbrace{(i\gamma_1\gamma_2)}_{\pm 1}} = \cos\frac{\pi}{4} + (i\gamma_1\gamma_2)\, i\sin\frac{\pi}{4}}$$

$$\begin{cases} U_{12}\gamma_1 U_{12}^\dagger = (U_{12})^2\gamma_1 = (i\gamma_1\gamma_2)\, i\gamma_1 = \gamma_2 \\ U_{12}\gamma_2 U_{12}^\dagger = (U_{12})^2\gamma_2 = (i\gamma_1\gamma_2)\, i\gamma_2 = -\gamma_1 \end{cases}$$

✓

Generalize to many domain walls / vortices:

$\overset{\times}{\gamma_1} \quad \overset{\times}{\gamma_2} \quad \overset{\times}{\gamma_3} \quad \cdots \quad \overset{\times}{\gamma_{p \in 2\mathbb{Z}}}$

braid

$\underbrace{— \quad — \quad — \quad \cdots \quad —}$

$2^{p/2}$ locally indistinguishable gd. states

$U_{j,j+1} = e^{i\frac{\pi}{4}(i\gamma_j\gamma_{j+1})}$ swaps γ_j, γ_{j+1}, sends

$$\boxed{|\psi_{init}\rangle \rightarrow U_{j,j+1}|\psi_{init}\rangle}$$

To check, label $|\psi_{init}\rangle$ by $i\gamma_{2j-1}\gamma_{2j}$ eigenvals, then compare $U_{j,j+1}|\psi_{init}\rangle$ to $|\psi_{init}\rangle$ w/ $\gamma_j \rightarrow \gamma_{j+1}$, $\gamma_{j+1} \rightarrow -\gamma_j$.

arbitrary initial gd. state

operator that rotates quantum state in degenerate manifold

Since $\boxed{[U_{j-1,j}, U_{j,j+1}] \neq 0}$, braiding is non-Abelian — final state depends on order of sequential swaps! Useful for...

Topological Qubits

Qubit requires 4 γ's due to fermion parity constraints.

$\overset{\times}{\gamma_1} \quad \overset{\times}{\gamma_2} \quad \overset{\times}{\gamma_3} \quad \overset{\times}{\gamma_4}$

$$\boxed{\begin{aligned} d_{12} &= \tfrac{1}{2}(\gamma_1 + i\gamma_2) \\ d_{34} &= \tfrac{1}{2}(\gamma_3 + i\gamma_4) \end{aligned}}$$

Even parity		Odd parity	
$\lvert 0_{12}\,0_{34}\rangle \equiv \lvert 0\rangle$	$\lvert 1_{12}\,1_{34}\rangle \equiv \lvert 1\rangle$	$\lvert 0_{12}\,1_{34}\rangle$	$\lvert 1_{12}\,0_{34}\rangle$

(assume relevant sector)

$$U_{12}|0\rangle = U_{34}|0\rangle = e^{-i\frac{\pi}{4}}|0\rangle$$

$$U_{12}|1\rangle = U_{34}|1\rangle = e^{+i\frac{\pi}{4}}|1\rangle$$

phase gate

$$U_{23}|0\rangle = \left[\cos\tfrac{\pi}{4} + (i\gamma_2\gamma_3)\, i\sin\tfrac{\pi}{4}\right]|0\rangle = \left[\cos\tfrac{\pi}{4} + (d_{12}-d_{12}^{+})(d_{34}+d_{34}^{+})\, i\sin\tfrac{\pi}{4}\right]|0\rangle$$

$$= \tfrac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$$U_{23}|1\rangle = \tfrac{1}{\sqrt{2}}\left(|1\rangle - i|0\rangle\right)$$

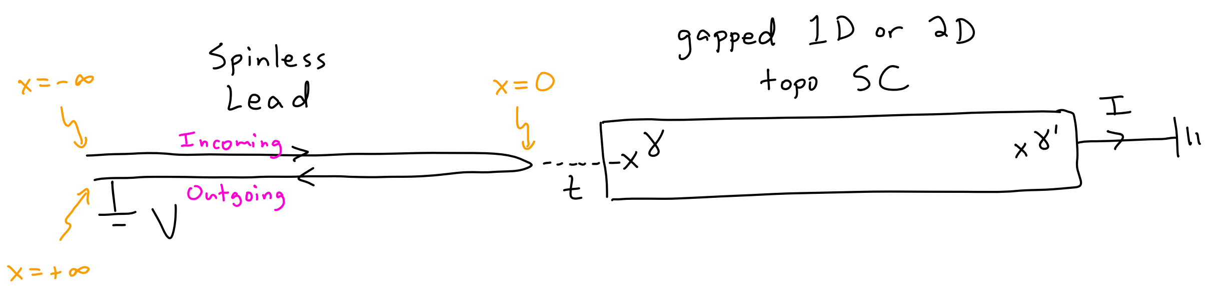
$\pi/2$ rot. on Bloch sphere

So braiding → "rigid" gates on fault-tolerant qubit space!

But how to detect zero modes / qubit states?

Tunneling Signatures of Majorana Zero Modes

Probes existence of Maj. modes via local measurement but not the qubits they encode. Setup:



Want sub-gap conductance (eV < bulk gap)

Freezes out normal transmission

$$\boxed{G(V) = \frac{e^2}{h} \times \left(2 P_{AR}\right)\Big|_{E=eV}}$$

Cooper pair factor

Andreev ref prob.

Model via

$$\boxed{\begin{aligned} H &= H_{lead} + H_t \\ H_{lead} &= \int_x \left(-i\hbar v\, \psi^\dagger \partial_x \psi\right) \\ H_t &= t\,\gamma\left[\psi(x=0) - \psi^\dagger(x=0)\right] \end{aligned}}$$

Only low-energy d.o.f. in SC!

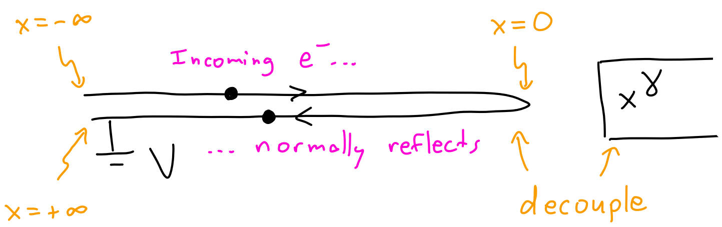
Dim. analysis: $P_{AR} = P_{AR}\left[|E|/\left(\frac{t^2}{\hbar v}\right)\right]$

Look @ $|E| \gg \frac{t^2}{\hbar v}$, $|E| \ll \frac{t^2}{\hbar v}$ regimes for simplicity.

- $|E| \gg \frac{t^2}{\hbar v}$

Here, $H \simeq H_{lead} \Rightarrow$ plane-wave eigensts. created by

$$\boxed{\Gamma_E^\dagger = \int_x e^{i\frac{E}{\hbar v}x}\,\psi^\dagger(x)}$$



$$\Rightarrow \boxed{G\left(|eV| \gg \frac{t^2}{\hbar v}\right) \simeq 0}$$

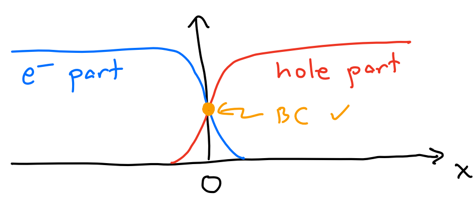
Not universal — additional terms, e.g., $\psi^\dagger \partial_x \psi^\dagger|_{x=0}$ could give finite G

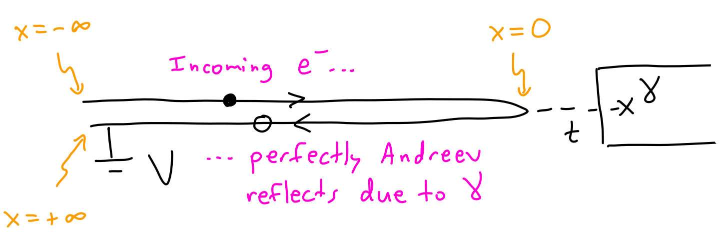
- $|E| \ll \frac{t^2}{\hbar v}$

"Large" H_t imposes $\boxed{\psi(x=0) = \psi^\dagger(x=0)}$ BC's!

$\Rightarrow$ plane-waves w/ $e^- \leftrightarrow$ hole conversion @ $x=0$,

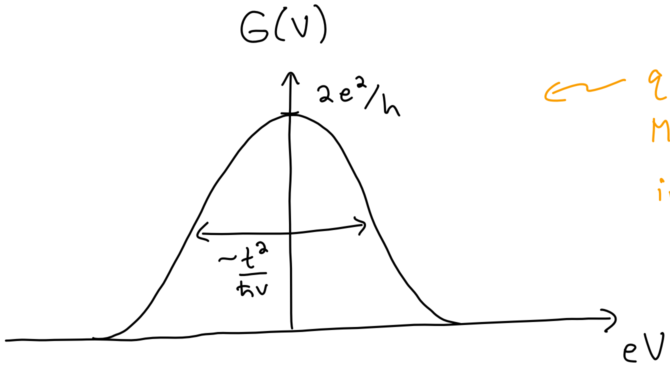
$$\boxed{\Gamma_E^\dagger = \int_x e^{i\frac{E}{\hbar v}x}\left[\Theta(-x)\,\psi^\dagger(x) + \Theta(x)\,\psi(x)\right]}$$





$$\Rightarrow \boxed{G\left(|eV| \ll \frac{t^2}{\hbar v}\right) \simeq \frac{2e^2}{h}}$$

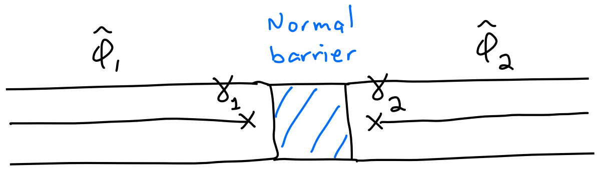
Universal!



← quantized zero bias peak – appealing Majorana signature, but delicate in practice

Topological SC Josephson Junctions

Provide qubit readout tool for 1D topo SC's.



← Kitaev chains promoted to # conserving formulation: $\Delta c_x^\dagger c_{x+1}^\dagger \to \Delta e^{i\hat{\varphi}} c_x^\dagger c_{x+1}^\dagger$

even/odd fermion # states degenerate $\Rightarrow$ single e^- tunneling not suppressed unlike conventional JJ's

$$\boxed{H_{JJ} = \underbrace{-E_J \cos(\hat{\varphi}_2 - \hat{\varphi}_1)}_{\text{2e CP hopping}} - E_J'(i\gamma_1\gamma_2)\underbrace{\cos\left(\frac{\hat{\varphi}_2 - \hat{\varphi}_1}{2}\right)}_{\text{hops charge e...}}}$$

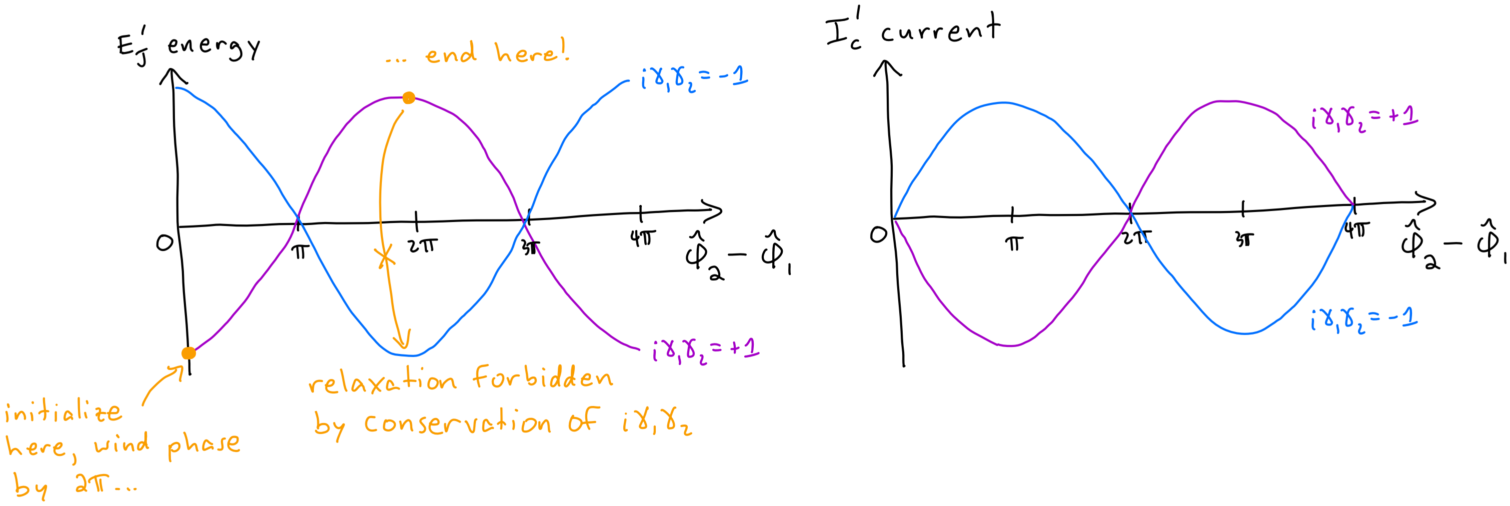
... flips fermion parities accordingly

Still 2π periodic in $\hat{\varphi}_{1,2}$ since $\hat{\varphi}_j \to \hat{\varphi}_j + 2\pi$, $\gamma_j \to -\gamma_j$ $(j = 1 \text{ or } 2)$ leaves H_{JJ} invariant. But current response is not since $i\gamma_1\gamma_2$ is a conserved quantity!

Using $\hat{I} = -\frac{2e}{\hbar}\frac{\partial H_{JJ}}{\partial(\hat{\varphi}_2 - \hat{\varphi}_1)}$ gives

$$\boxed{I = -I_c \sin(\hat{\varphi}_2 - \hat{\varphi}_1) - I_c'(i\gamma_1\gamma_2)\sin\left(\frac{\hat{\varphi}_2 - \hat{\varphi}_1}{2}\right)}$$

Look @ E_J', I_c' pieces:



With fixed $i\gamma_1\gamma_2$, energy/current are 4π periodic in phase diff!
("Fractional Josephson Effect")

but "quasiparticle poisoning" is a practical issue...

anomalous pumping cycle akin to flux threading in FQH

$$\boxed{\text{Measure current} \rightarrow \text{infer } i\gamma_1\gamma_2}$$